Shanghai University of Finance and Economics

Entrance Examination for 2021 Interdisciplinary Sciences Elite Program

Date: September 03, 2021
Time: 6:00pm – 10:00pm

Background

"Mystery boxes" (盲盒) are gaining great popularity in China in recent years. A mystery box is a special product that contains random goods (随机商品) in it. The buyer does not know what is inside the box at the time of purchase. Such a product is often called a *probabilistic good* (概率商品).

A probabilistic good offers a probability of getting any one of a set of items. We use *probabilistic selling* (概率销售) to denote (表示) the selling strategy (销售策略) under which the seller creates probabilistic goods using the seller's existing products or services and offers such probabilistic goods to potential buyers as additional purchase choices. For example, a retailer selling two different colors of sweaters, red and green, may offer an additional "probabilistic sweater," which can be either the red or green sweater. We use the term *traditional selling* (传统销售) to denote the conventional selling strategy under which the seller only offers the existing goods (i.e., no probabilistic good) for sale.

Our interest is to compare the two selling strategies and explore the fundamental product/market conditions required for the benefit of introducing uncertainty in product assignments by offering "probabilistic goods".

Traditional Selling Strategy

We start with the analysis of the traditional selling strategy. Consider a firm offering two component goods, $j = 1, 2$, which have the same production costs (生产成本): $c_1 = c_2 = c$, $0 < c < 1$. We assume that the seller is aware of the demand (需求) and able to satisfy all demand (if it so desires). Under the traditional selling strategy, the firm sells each good j at a price p_j. We assume that $0 \le p_j \le 1$ for $j = 1, 2$. The firm's objective is to determine the prices of the goods to maximize (最大化) its total profit (利润).

Let v_{ji} be the value of good j to consumer (消费者) i. Different consumers have different tastes. Some may like good 1 more than good 2, but others may like good 2 more. To describe consumers' different tastes, we assume all consumers are uniformly

(均匀地) located on the interval (区间) [0,1], good 1 is located at point 0, and good 2 is located at point 1 (see Figure 1). A consumer likes a good more if she is closer to this good. Let x_i denote consumer i's location. The *valuations* (评分值) of two goods to consumer i are

$$\begin{cases} v_{1i} = 1 - x_i, \\ v_{2i} = x_i, \end{cases} \qquad \text{where } x_i \in [0,1]. \tag{1}$$

For example, as shown in Figure 1, if consumer i's location in the interval is $1/2$, i.e., $x_i = 1/2$, then her valuation for both good 1 and good 2 are $1/2$. If her location is at $1/4$, i.e., $x_i = 1/4$, then her valuation for good 1 and good 2 are respectively (分别是) $3/4$ and $1/4$. In this case, because the consumer is located closer to good 1, her valuation for good 1 is higher.

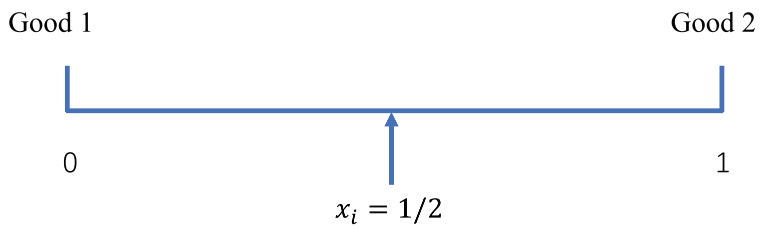

图 1: Consumer valuation

We assume each consumer buys at most one good. The consumer's *utility* (效用值) from buying a good is the difference between her valuation and the price of this good. For example, the utility of consumer i from buying good 1 (or good 2) under the traditional selling strategy is $v_{1i} - p_1$ (or $v_{2i} - p_2$).

Under the traditional selling strategy, each consumer has three choices: (a) buy good 1, (b) buy good 2, and (c) buy nothing. She chooses the one that leads to the highest utility. More specifically, consumer i will buy good 1 if

$$v_{1i} - p_1 \geq v_{2i} - p_2 \text{ and } v_{1i} - p_1 \geq 0;$$

and she will buy nothing if

$$v_{1i} - p_1 < 0 \text{ and } v_{2i} - p_2 < 0.$$

Next, we derive (推导) the demand function (需求函数) which shows the relationship between the prices of two goods and the sizes of consumers buying the goods at those prices. Recall that consumers are uniformly located in interval [0,1]. Because the shorter is the distance between a consumer and a good, the higher is the valuation of the consumer from buying this good, given the prices of two goods, consumers will be divided into at most three segments (区间的分割) as shown in Figure 2. In particular, consumers in the left segment $[0,\hat{x}_1]$ buy good 1, consumers in the right segment $[\hat{x}_2,1]$ buy good 2, and consumers in the middle segment $(\hat{x}_1, \hat{x}_2)$ buy nothing.

Let $D_1(p_1, p_2)$ represent the demand function for good 1 under the traditional selling strategy ($D_1(p_1, p_2)$是一个自变量为p_1, p_2的二元函数). It is the length of segment $[0, \hat{x}_1]$ (as you know, this is simply $\hat{x}_1$). Let $D_2(p_1, p_2)$ represent the demand function for good 2, which is the length of segment $[\hat{x}_2, 1]$ (this is simply $1 - \hat{x}_2$). Therefore, to determine $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$, we need to figure out the thresholds (分界点) $\hat{x}_1$ and $\hat{x}_2$.

Question 1 (15 pts): Show that, under the traditional selling strategy, the demand for good 1 is given by

$$D_1(p_1, p_2) = \begin{cases} 1 - p_1 & \text{if } p_1 + p_2 \geq 1, \\ \dfrac{1 - p_1 + p_2}{2} & \text{if } p_1 + p_2 < 1. \end{cases}$$

(Hint: for the consumer at location $\hat{x}_1$, she should be indifferent (无区别的) between buying good 1 and not buying good 1, i.e., buying good 2 or buying nothing. Mathematically, this is equivalent to $1 - \hat{x}_1 - p_1 = \max\{0, \hat{x}_1 - p_2\}$. You can consider two cases: (1) $1 - \hat{x}_1 - p_1 = 0 \geq \hat{x}_1 - p_2$, (2) $1 - \hat{x}_1 - p_1 = \hat{x}_1 - p_2 > 0$. Note that $\hat{x}_1$ is within the interval [0,1].)

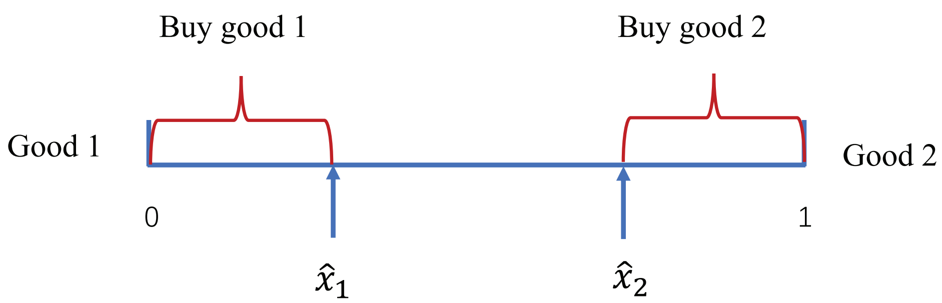

图 2: Consumer segments

Answer 1: In case (1) we get from $1 - \hat{x}_1 - p_1 = 0$ that $\hat{x}_1 = 1 - p_1$. Then, from $\hat{x}_1 - p_2 \leq 0$ we further get $p_1 + p_2 \geq 1$. In case (2), we get from $1 - \hat{x}_1 - p_1 = \hat{x}_1 - p_2$ that $\hat{x}_1 = (1 - p_1 + p_2)/2$. Then from $\hat{x}_1 - p_2 \geq 0$ we get $p_1 + p_2 < 1$. The demand for good 1 is the length of the segment $[0, \hat{x}]_1$, so $D_1(p_1, p_2) = \hat{x}_1$.

Let $F(p_1, p_2)$ denote the profit function of the firm. Next, we determine the optimal (最优的) prices that maximize $F(p_1, p_2)$, which is given by

$$F(p_1, p_2) = \sum_{j=1}^{2} (p_j - c) \cdot D_j(p_1, p_2) \tag{2}$$

Assume good 2 has the same demand as good 1, $D_2(p_1, p_2) = D_1(p_1, p_2)$. Here $D_1(p_1, p_2)$ is the expression (表达式) in Question 1. Also assume the seller always set the same price for the two goods $p_1 = p_2 = p$. Then we can rewrite the two demand functions as $\bar{D}(p) = D_1(p, p)$ and the profit function as $\bar{F}(p) = F(p, p)$.

Question 2 (15 pts): Write down the demand function $\bar{D}(p)$ and the profit function $\bar{F}(p)$. Find the optimal price p^* (that is, the value of p (p的取值) that maximizes $\bar{F}(p)$) and the corresponding (相应的) optimal profit $F^* = \bar{F}(p^*)$.

Answer 2: When $p_1 = p_2 = p$, we know that

$$\bar{D}(p) = D_1(p, p) = \begin{cases} 1 - p & \text{if } p \geq 1/2, \\ 1/2 & \text{if } p < 1/2. \end{cases}$$

Then we can get

$$\bar{F}(p) = 2(p - c)\bar{D}(p) = \begin{cases} 2(p - c)(1 - p) & \text{if } p \geq 1/2, \\ p - c & \text{if } p < 1/2. \end{cases}$$

When $p \geq 1/2$, the optimal price is $(1 + c)/2$ (which is guaranteed to be $> 1/2$ since $c > 0$), and the optimal profit is $(1 - c)^2/2$. When $p < 1/2$, the optimal price is $1/2$, and optimal profit is $\frac{1}{2} - c$). Clearly $(1 - c)^2/2 > \frac{1}{2} - c$, so the final optimal price is $p^* = (1 + c)/2$, and optimal profit is $F^* = (1 - c)^2/2$.

Probabilistic Selling Strategy

Next, we study the probabilistic selling strategy. In this case, the firm sells each component good j at a price q_j ($j = 1, 2$). The firm also sells at price q_0 a probabilistic good, which has probability $1/2$ to be component good 1 and probability $1/2$ to be component good 2. We assume that $0 \leq q_j \leq 1$ for $j = 0, 1, 2$.

Suppose consumer i's valuation for the probabilistic good is

$$v_{0i} = \frac{1}{2}v_{1i} + \frac{1}{2}v_{2i},$$

her utility from buying good 1 (or good 2) is $v_{1i} - q_1$ (or $v_{2i} - q_2$), and her utility from buying the probabilistic good is $v_{0i} - q_0$.

Under the probabilistic selling strategy, consumer i has four choices: (a) buy product 1, (b) buy product 2, (c) buy the probabilistic good, and (d) buy nothing. She chooses the one that has the highest utility. For example, consumer i will buy product 1 if

$$v_{1i} - q_1 \geq v_{2i} - q_2 \text{ and } v_{1i} - q_1 \geq v_{0i} - q_0 \text{ and } v_{1i} - q_1 \geq 0;$$

and she will buy nothing if

$$v_{1i} - q_1 < 0 \text{ and } v_{2i} - q_2 < 0 \text{ and } v_{0i} - q_0 < 0.$$

To simplify the analysis, we assume that the firm always sets the same price for good 1 and good 2, that is, $q_1 = q_2 = q$.

Question 3 (15 pts): Prove that, if $q_0 > q$, then nobody will buy the probabilistic good. (Hint: Analyze the valuation of a consumer from buying the probabilistic good and compare it with the valuation from buying either good 1 or good 2.)

Answer 3: When $q_0 > q$, we have

$$\frac{1}{2}(1 - x_i) + \frac{1}{2}x_i - q_0 < \frac{1}{2}(1 - x_i - q) + \frac{1}{2}(x_i - q)$$

for any $x_i \in [0, 1]$. That is, the utility of probabilistic good is less than the average of the utilities of the two component goods. Therefore, utility of probabilistic good has to be less than either one of the utilities of the two component goods

From Question 3, we know that if the seller decides to offer the probabilistic good, she must set $q_0 \leq q$. If $q_0 \leq q$, consumers located close to $\frac{1}{2}$ are likely to buy the probabilistic good. Thus consumers will be divided into at most five segments as shown in Figure 3. In particular, consumers in the left segment [0,$\hat{x}_1$] buy good 1, consumers in the right segment [$\hat{x}_2$,1] buy good 2, and consumers in the middle segment $[\hat{x}_3, \hat{x}_4]$ buy the probabilistic good.

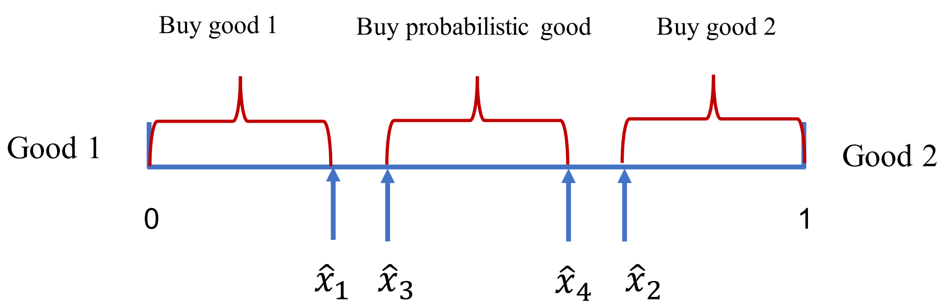

图 3: Consumer segments

Assume demand for good 1 and good 2 are the same. Let $E(q, q_0)$ represent the demand function for good 1 or good 2, and $E_0(q, q_0)$ represent the demand for the probabilistic good. To determine these demand functions, we need to figure out the thresholds $\hat{x}_1 \sim \hat{x}_4$.

Question 4 (15 pts): Show that, under the probabilistic selling strategy, if $q_0 \le \frac{1}{2}$, then $\hat{x}_3 = \hat{x}_1$, $\hat{x}_4 = \hat{x}_2$; otherwise if $q_0 > \frac{1}{2}$, then $E_0(q, q_0) = 0$.

Answer 4: For consumer i, the utility of buying good 1, good 2 and the probabilistic products are $1 - x_i - q$, $x_i - q$ and $1/2 - q_0$ respectively. She chooses the probabilistic product if and only if

$$\frac{1}{2} - q_0 \ge 1 - x_i - q \text{ and } \frac{1}{2} - q_0 \ge x_i - q \text{ and } \frac{1}{2} - q_0 \ge 0.$$

When $q_0 > 1/2$, the third inequality cannot hold, so $E_0(q, q_0) = 0$.

When $q_0 \le 1/2$, the first two inequalities give

$$\frac{1}{2} - (q - q_0) \le x_i \le \frac{1}{2} + (q - q_0).$$

When $x_i = \frac{1}{2} - (q - q_0)$, the consumer is indifferent between good 1 and the probabilistic good (their utilities are equal). When $x_i = \frac{1}{2} + (q - q_0)$, the consumer is indifferent between good 2 and the probabilistic good. So we must have $\hat{x}_3 = \hat{x}_1$, $\hat{x}_4 = \hat{x}_2$, and $E_0(q, q_0) = 2(q - q_0)$.

Questions 3 and 4 tell us that, if the seller decides to offer the probabilistic good, she must set $q_0 = \frac{1}{2} < q$, in which case consumers are divided into three segments; otherwise, the seller won't offer the probabilistic good, which becomes the traditional selling case. Next, we determine the price q to maximize the profit

$$G(q, q_0) = 2(q - c) \cdot E(q, q_0) + (q_0 - c) \cdot E_0(q, q_0),$$

where $q_0 = \frac{1}{2}$.

Question 5 (15 pts): Given $q_0 = \frac{1}{2}$, find the optimal price q^* (that is, the value of q that maximizes $G(q, q_0)$) and optimal profit $G^* = G(q^*, q_0)$.
(<u>Hint</u>: Similar to the analysis of the traditional selling strategy, find out the thresholds $\hat{x}_1$ and $\hat{x}_2$ first.)

Answer 5: As shown in Question 4, the thresholds are $\hat{x}_1 = \hat{x}_3 = \frac{1}{2} - (q - q_0)$, $\hat{x}_2 = \hat{x}_4 = \frac{1}{2} + (q - q_0)$. Then we know that $E(q, q_0) = \frac{1}{2} - (q - q_0)$ and $E_0(q, q_0) = 2(q - q_0)$. Therefore,

$$\begin{aligned} G(q, q_0) &= (q - c)\{1 - 2(q - q_0)\} + 2(q_0 - c)(q - q_0) \\ &= q - c - 2(q - q_0)^2. \end{aligned}$$

When $q_0 = 1/2$, the optimal q is $q^* = q_0 + \frac{1}{4} = \frac{3}{4}$, and the optimal profit is $G^* = q_0 - c + \frac{1}{8} = \frac{5}{8} - c$.

Question 6 (15 pts): Compare F^* and G^* that you have obtained, and find the conditions under which the seller will adopt (采用) the probabilistic selling strategy and offer the probabilistic good.

Answer 6: By comparing $F^* = (1-c)^2/2$ and $G^* = \frac{5}{8} - c$, we get that $G^* > F^*$ if and only if $0 < c < 1/2$.

Question 7 (10 pts): Based on the above analysis, can you give some explanation about why sellers offer probabilistic goods like "mystery boxes" to the market? Besides the above analysis, can you list some other advantages of probabilistic selling?