Shanghai University of Finance and Economics

Entrance Examination for 2023 Interdisciplinary Sciences Elite Program

Date: September 01, 2023

Time: 6:00pm – 9:00pm

There are 10 questions, and total score is 100.

1 Background

Inventory management (库存管理) is the process of planning and controlling the flow of goods in and out of a business. It helps to ensure that the business has enough products to meet customer demand, while avoiding the waste and expense of overstocking or understocking. Inventory management involves deciding when to order new products, how many to order, and how to store them efficiently. Suppose you own a bookstore that sells books, magazines, and stationery (文具). You want to keep track of how many items you have in stock (存货), how much they cost, and how fast they sell. You also want to avoid the problems of overstocking or understocking your products. If you store too few products, you may run out of stock (缺货) and lose sales, profits, and competitive edge. If you store too many products, you may waste money and space, and risk having unsold or damaged goods.

One common inventory problem is how to replenish (补充) inventory when it is depleted by customer demand. This problem can be modeled by assuming that the demand rate (单位时间需求量) for a product is constant and known, denoted by R, and that the product is ordered in batches. The costs involved in this problem are: (1). the cost of holding each unit of product in inventory per unit time, C_1; (2). the cost of being unable to meet each unit of customer demand per unit time, C_2; (3). the fixed cost of placing an order for a batch of (一批) products, C_3; (4). the cost of producing or purchasing each unit of product, K.

The goal of inventory management is to find the best ordering policy that minimizes the total cost per unit time, while satisfying customer demand. This policy specifies when to order a new batch of products, and how many units to order each time.

2 Inventory replenishment with zero shortage and instant delivery

To simplify the problem, we first assume that we never run out of stock. In other words, we set the shortage cost, C_2, to be very high. We also assume that we can order a batch of products and receive them instantly when our inventory level reaches zero. This way, we can avoid any delays or uncertainties in the supply chain.

Let t be the time interval between two consecutive orders, and Q be the quantity of products ordered each time. Figure 1 shows how the inventory level changes over time under this policy. We start with an inventory level of zero, and order Q units at time zero. Then, we sell the products at a constant rate of R units per unit time, until the inventory level drops to zero again. At this point, we order another batch of Q units, and repeat the cycle.

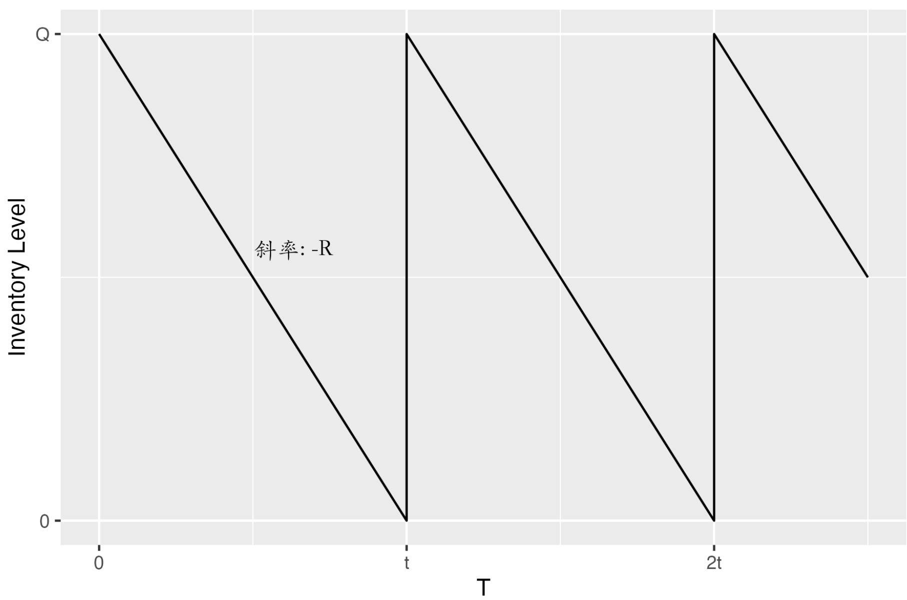

图 1: Inventory level over time when we order and receive products immediately.

We know the values of R, C_1, C_3 and K, which are the demand rate, the holding cost per unit product per unit time, the fixed cost per order, and the unit cost per product, respectively. Our goal is to find the optimal value of Q, which minimizes the total cost per unit time.

To help you understand the notations more clearly, we give you an example of how to calculate the total cost per cycle. The production or ordering cost per cycle is $C_3 + KQ$. The holding cost per cycle is $C_1Qt/2 = C_1Q^2/2/R$. Therefore, the total cost per cycle is

$C_3 + KQ + C_1Q^2/2/R$.

Question 1. What is the expression for the total cost per **unit time**, denoted by $C(Q)$, as a function of Q?

Answer 1: The total cost per unit time is

$$C(Q) = \frac{C_3 + KQ}{t} + C_1Q/2 = \frac{C_3 + KQ}{Q/R} + \frac{C_1Q}{2} = \frac{C_3R}{Q} + KR + \frac{C_1Q}{2}.$$

Now we determine when and by how much to replenish inventory through minimizing the total cost per **unit time**.

Question 2. What is the optimal value of Q, denoted by Q^*, that minimizes $C(Q)$? What is the corresponding value of t, denoted by t^*?

Answer 2: The optimal value of Q is given by

$$Q^* = \sqrt{\frac{2C_3R}{C_1}}.$$

The corresponding t^* is $t^* = Q^*/R = \sqrt{\frac{2C_3}{C_1R}}$.

3 Inventory replenishment with stockouts and backorders

Sometimes, it may be beneficial to allow some stockouts (缺货), or planned shortages, in inventory management. This means that we are willing to accept a delay in fulfilling some customer orders, if the holding cost of inventory (C_1) is too high compared to the shortage cost (C_2). The shortage cost is the cost of losing customer satisfaction or loyalty due to unavailability of products.

When we have stockouts, we keep track of the customer orders that are not met, and we call them backorders. Backorders are orders that are placed by customers but cannot be delivered immediately because the product is out of stock. The business promises to deliver the product as soon as possible, but the customer has to wait until then. This can affect the customer's perception of the business and its service quality. We fill the backorders as soon as we receive a new batch of products. In this case, the inventory level over time looks like Figure 2. It shows that when the inventory level reaches zero, we do not order immediately, but wait for some time, t_1, until we order a batch of Q

units. During this time, we accumulate backorders, which are filled when the new batch arrives. Then we sell the remaining at a constant rate of R units per unit time. **Note that there is no holding cost during the period** $(0, t_1)$**.**

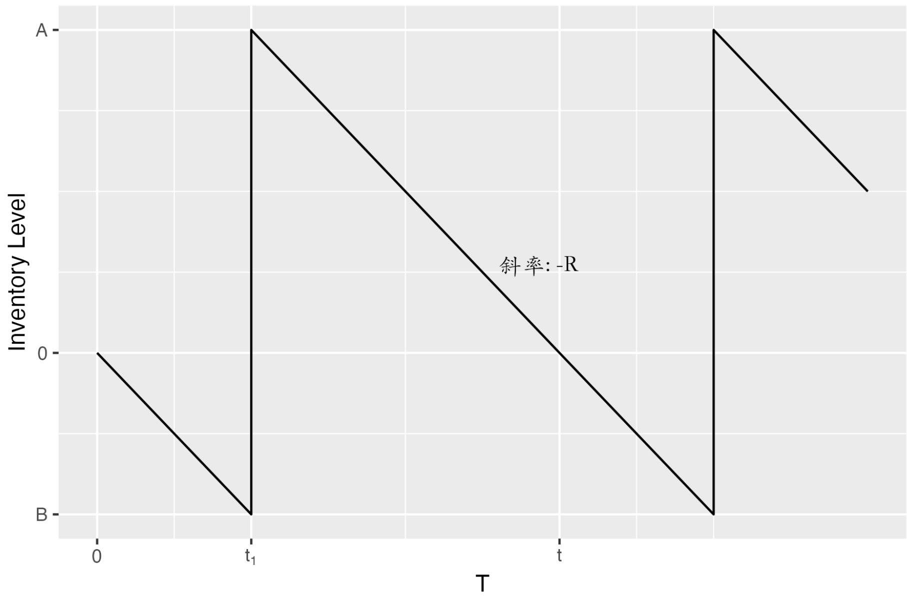

图 2: Inventory level over time when we allow some stockouts and backorders.

We know the values of R, C_1, C_2, C_3 and K, which are the demand rate, the holding cost per unit product per unit time, the shortage cost per unit product per unit time, the fixed cost per order, and the unit cost per product, respectively. Our goal is to find the optimal values of t_1 and Q, which minimize the total cost per unit time.

Question 3. What is the expression for the total cost per **unit time**, denoted by $C(t_1, Q)$, as a function of t_1 and Q?

Answer 3: The production or ordering cost per cycle $C_3 + KQ$. The holding cost per cycle is $C_1 * A/2 * (t - t_1) = C_1 * R * (t - t_1)^2/2 = C_1 * R * (Q/R - t_1)^2/2$. The shortage cost per cycle is $C_2 * B/2 * t_1 = C_2 * R * t_1^2/2$. Therefore, the total cost per unit time is

$$\begin{aligned} C(t_1, Q) &= \frac{C_3 + KQ}{t} + \frac{C_1 R(Q/R - t_1)^2}{2t} + \frac{C_2 R t_1^2}{2t} \\ &= \frac{C_3 R}{Q} + KR + \frac{C_1 (Q - t_1 R)^2}{2Q} + \frac{C_2 R^2 t_1^2}{2Q}. \end{aligned}$$

Now we minimize the total cost per **unit time** to determine when and by how much to replenish inventory.

Question 4. For each fixed Q, what is the optimal value of t_1, denoted by $t_1^*(Q)$, that minimizes $C(t_1, Q)$?

Answer 4: For each Q, the optimal t_1 is given by

$$t_1^*(Q) = \frac{C_1 Q}{(C_1 + C_2)R}.$$

Question 5. What are the optimal values of t_1 and Q, denoted by t^* and Q^*, that minimize $C(t_1, Q)$?

Answer 5: Substituting $t_1^*(Q) = \frac{C_1 Q}{(C_1+C_2)R}$ into $C(t_1^*(Q), Q)$, we have,

$$\begin{aligned} C(t_1^*(Q), Q) &= \frac{C_3 R}{Q} + KR + \frac{C_1(Q - \alpha Q)^2}{2Q} + \frac{C_2 \alpha^2 Q^2}{2Q} \\ &= \frac{C_3 R}{Q} + KR + \frac{C_1(1-\alpha)^2 Q}{2} + \frac{C_2 \alpha^2 Q}{2}, \end{aligned}$$

where $\alpha = C_1/(C_1 + C_2)$. Then the optimal Q is given by

$$Q^* = \sqrt{\frac{2C_3 R(C_1 + C_2)}{C_1 C_2}}.$$

The corresponding t^* is

$$t^* = \sqrt{\frac{2C_1 C_3}{RC_2(C_1 + C_2)}}.$$

4 Inventory replenishment with finite production rate

Sometimes, we cannot order a batch of products and receive them instantly. Instead, we need to produce the products ourselves, or wait for the supplier to produce them for us. We assume that we can start the production process whenever we want, but it takes some time to complete. We also assume that we can store the products in our inventory as soon as they are produced, without any delay in transportation. Let P be the constant production rate, which is higher than R. In this case, the inventory level over time looks like Figure 3.

We know the values of R, P, C_1, C_2, C_3 and K, which are the demand rate, the production rate, the holding cost per unit product per unit time, the shortage cost per

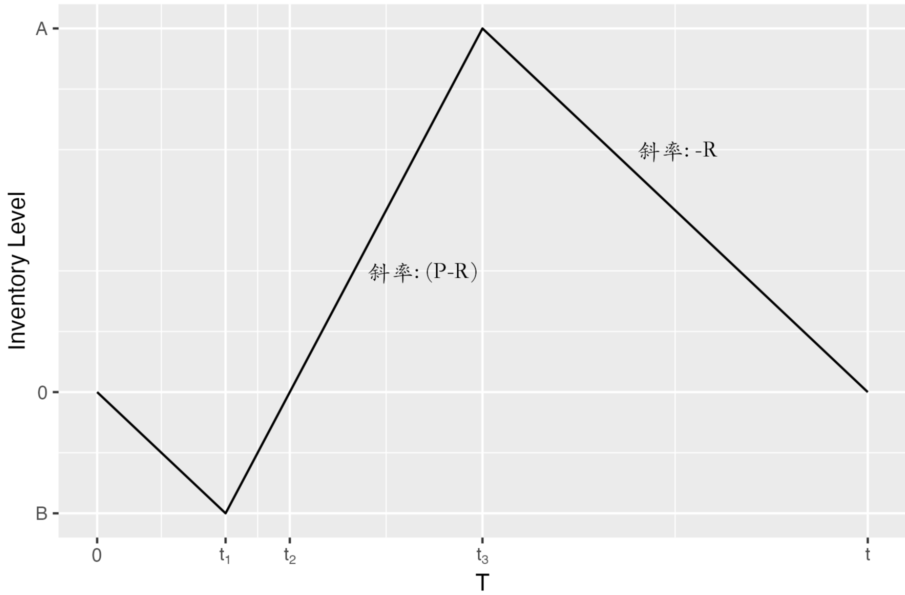

图 3: Inventory level over time when we have a delay in producing or receiving products.

unit product per unit time, the fixed cost per order, and the unit cost per product, respectively. Our goal is to find the optimal values of t_1 and Q, which minimize the total cost per unit time.

Question 6. What is the expression for the total cost per **unit time**, denoted by $C(t_1, Q)$, as a function of t_1 and Q?

Answer 6: First of all, $Q = Rt$, $B = Rt_1 = (P-R)(t_2-t_1)$, $A = R(t-t_3) = (P-R)(t_3-t_2)$. Then we have $t = Q/R$, $t_2 = Pt_1/(P-R)$, and $t_3 = t_1 + R/Pt = t_1 + Q/P$.

The production or ordering cost per cycle $C_3 + KQ$. The holding cost per cycle is $C_1 * A/2 * (t - t_2) = C_1 * (Q - Rt_1 - RQ/P) * (Q/R - Pt_1/(P-R))/2$. The shortage cost per cycle is $C_2 * B/2 * t_2 = C_2 * P * R * t_1^2/(P-R)/2$. Therefore, the total cost per unit time is

$$\begin{aligned} C(t_1, Q) &= \frac{C_3 + KQ}{t} + \frac{C_1(Q - Rt_1 - RQ/P)(Q/R - Pt_1/(P-R))}{2t} + \frac{C_2PRt_1^2}{2t(P-R)} \\ &= \frac{C_3R}{Q} + KR + \frac{C_1(Q-\beta Rt_1)^2}{2\beta Q} + \frac{C_2\beta R^2t_1^2}{2Q}, \end{aligned}$$

where $\beta = P/(P-R)$.

Now we minimize the total cost per **unit time** to determine when and by how much to replenish inventory.

Question 7. For each fixed Q, what is the optimal value of t_1, denoted by $t_1^*(Q)$, that minimizes $C(t_1, Q)$?

Answer 7: For each Q, the optimal t_1 is given by

$$t_1^*(Q) = \frac{C_1 Q}{\beta(C_1 + C_2)R}.$$

Question 8. What are the optimal values of t_1 and Q, denoted by t^* and Q^*, that minimize $C(t_1, Q)$?

Answer 8: Substituting $t_1^*(Q) = \frac{C_1 Q}{\beta(C_1+C_2)R}$ into $C(t_1^*(Q), Q)$, we have,

$$\begin{aligned} C(t_1^*(Q), Q) &= \frac{C_3 R}{Q} + KR + \frac{C_1(Q - \alpha Q)^2}{2\beta Q} + \frac{C_2 \alpha^2 Q}{2Q\beta} \\ &= \frac{C_3 R}{Q} + KR + \frac{C_1(1-\alpha)^2 Q}{2\beta} + \frac{C_2 \alpha^2 Q}{2\beta}, \end{aligned}$$

where $\alpha = C_1/(C_1 + C_2)$. Then the optimal Q is given by

$$Q^* = \sqrt{\frac{2C_3 R(C_1 + C_2)}{C_1 C_2}} \sqrt{\frac{P}{P-R}}.$$

The corresponding t^* is

$$t^* = \sqrt{\frac{2C_1 C_3}{RC_2(C_1 + C_2)}} \sqrt{\frac{P}{P-R}}.$$

5 Inventory management with stochastic demand

So far, we have assumed that the demand for the product is constant and known in advance. However, in reality, the demand may vary from day to day, and we may not be able to predict it accurately. In this case, we say that the demand is stochastic, meaning that it is random and follows a certain probability distribution.

For simplicity, we only consider a single-period inventory problem, such as a newsvendor who sells newspapers every day. The newsvendor needs to decide how many newspapers to order before knowing the actual demand for that day, and then sell them at a fixed price per unit. At the end of the day, the newsvendor earns a profit of k dollars for each newspaper sold, and incurs a cost of h dollars for each newspaper left unsold.

The newsvendor faces a trade-off between ordering too many newspapers and having excess inventory, or ordering too few newspapers and missing potential sales. The goal is to find the optimal order quantity, Q^*, that maximizes the expected profit per day.

Let R be the random variable that represents the demand for newspapers per day. Let $\mathrm{P}(r)$ be the probability that the demand is equal to r, where $r = 0, 1, \ldots$. We assume that $\sum_{r=0}^{\infty} \mathrm{P}(r) = 1$.

Question 9. What is the expected revenue if the order quantity is Q? (Hint: Use the formula for the expected value of a function of a random variable: $E[f(R)] = \sum_{r=0}^{\infty} f(r)\mathrm{P}(r)$.)

When the demand is r, the revenue is $kr - h(Q-r)$ if $r \leq Q$ and kQ if $r > Q$. Therefore, the expected revenue is

$$\begin{aligned}
&\sum_{r=0}^{Q}\{kr - h(Q-r)\}\mathrm{P}(r) + \sum_{r=Q+1}^{\infty} kQ\mathrm{P}(r) \\
=\ &\sum_{r=0}^{Q}\{kr - h(Q-r)\}\mathrm{P}(r) + \sum_{r=Q+1}^{\infty}\{k(Q-r)\}\mathrm{P}(r) + \sum_{r=Q+1}^{\infty} kr\mathrm{P}(r) \\
=\ &kE(R) - \sum_{r=Q+1}^{\infty}\{k(r-Q)\}\mathrm{P}(r) - \sum_{r=0}^{Q}\{h(Q-r)\}\mathrm{P}(r).
\end{aligned}$$

Question 10. Define $F(Q) = \sum_{r=0}^{Q} \mathrm{P}(r)$. This is the probability that the demand is less than or equal to Q. Show that the optimal order quantity, Q^*, can be determined by finding the value of Q such that

$$F(Q-1) < \frac{k}{k+h} \leq F(Q).$$

Let $C(Q)$ be the difference between $kE(R)$ and the expected revenue when the order quantity is Q. That is,

$$C(Q) = \sum_{r=Q+1}^{\infty}\{k(r-Q)\}\mathrm{P}(r) + \sum_{r=0}^{Q}\{h(Q-r)\}\mathrm{P}(r).$$

Then we have

$$C(Q+1) - C(Q) = (k+h)\left[F(Q) - \frac{k}{k+h}\right].$$

This implies that when $F(Q) < \frac{k}{k+h}$, $C(Q+1) < C(Q)$, while when $F(Q) \geq \frac{k}{k+h}$, $C(Q+1) \geq C(Q)$. Therefore, when $F(Q^*-1) < \frac{k}{k+h} \leq F(Q^*)$, $C(Q^*)$ attains the minimum of $C(Q)$, or equivalently, the revenue attains the maximum.